Sensitivity Analysis of the Solutions of Perturbed Linear Systems

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Abstract

This study investigated the sensitivity of the solutions of perturbed linear systems of ordinary differential equations. To achieve this, the concepts of matrix norms, condition numbers and conditioning were first explored. Sequel to this, we demonstrated the validity of perturbation theory by applying it to solve an algebraic equation and a standard second order ordinary differential equation, namely, the harmonic oscillator equation, and comparing the results to their corresponding exact solutions. We then examined the effects of small perturbations in the coefficient matrices of linear systems of algebraic and differential equations to the solution vectors. The idea of matrix norms was applied to obtain the condition numbers which were in turn used to measure the sensitivity of the linear systems. We found that the systems whose associated matrices have large condition numbers are usually very sensitive to small perturbations while the reverse is the case for low condition numbers.

Keywords: Sensitivity of solutions, matrix norm, perturbation theory, harmonic oscillator, condition number, conditioning.

1 Introduction

Perturbation theory is an approach to finding an approximate solution to a problem, by starting from the exact solution of a related, simpler problem [5]. It uses the fact that several complicated real life problems which are not exactly solvable contain some small parameter(s) ε (say), which can be set to zero to make the equations exactly solvable, and that the approximate solutions of these complicated problems can be obtained from the exact solutions of the ideal ones. This method was first proposed for the solution of problems in celestial mechanics, in the context of the motions of planets in the solar system [1]. It was later extended and generalized by several notable 18th and 19th century mathematicians such as Lagrange and Laplace as a result of its incremental demand in the accuracy of solutions to Newton's gravitational equations [2].

Recently, perturbation theory has been gaining much popularity and it is a very broad subject with applications in many areas of the physical sciences, see e.g. [4], [7] and [8]. It is successful in solving a wide range of both algebraic and differential equations analytically. Consider, for instance, the problem of approximating the perturbed equation

$$x^2 - 3x + 2 + \varepsilon = 0 \tag{1.1}$$

which depends on a small parameter ε, $0 < \varepsilon \ll 1$, from the exact solution of the unperturbed equation

$$x^2 - 3x + 2 = 0 \tag{1.2}$$

obtained by setting $\varepsilon = 0$ in (1.1).

Consider also the problem of approximating the damped harmonic oscillator equation given by

$$\ddot{x} + \varepsilon\dot{x} + x = 0 \tag{1.3}$$

from the exact solution of the undamped harmonic oscillator given by

$$\ddot{x} + x = 0 \tag{1.4}$$

obtained by setting $\varepsilon = 0$ in (1.3).

Again, consider the problem of finding an approximate solution to the perturbed linear system of first order ordinary differential equations given by

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} \varepsilon & 4 \\ 5 & -0.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1.5}$$

from the unperturbed system given by

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 5 & -0.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1.6}$$

obtained by setting $\varepsilon = 0$ in (1.5).

Now, the question is:
For small ε, $0 < \varepsilon \ll 1$,

- "How well does (1.2) approximate (1.1)?"
- "How well does (1.4) approximate (1.3)?"
- "How far do our solutions of the perturbed problem (1.5) go from the exact solutions of the unperturbed one given by (1.6)?" In other words, "How sensitive are the solutions to small perturbations in the coefficient matrix?"
- "Can we apply the idea of matrix norms to measure the sensitivity of solutions of systems of equations to small perturbations in the coefficient matrices?"

It is these questions that this study is poised to provide answers to.

2 Matrix norms, condition number and conditioning

2.1 Matrix norms

Definition 2.1. *[9, 11]. A matrix norm, $||\cdot||$ is a function $||\cdot|| : \mathbb{C}^{m\times n} \to \mathbb{R}$ such that for all $A, B \in \mathbb{C}^{m\times n}$ and $\alpha \in \mathbb{C}$ we have*

(1a) $||A|| \geq 0$ Nonnegativity

(1b) $||A|| = 0$ if and only if $A = 0$ Positivity

(2) $||\alpha A|| = |\alpha|\ ||A||$ Homogeneity

(3) $||A + B|| \leq ||A|| + ||B||$ Triangle Inequality (subadditivity).

The most frequently used matrix norms are the Frobenius norm and the *p-norms*, see e.g. [6].

The *Frobenius*(*Schur* or *Hilbert-Schmidt*) norm[9]

The *Frobenius norm* $||A||_F$ is defined such that

$$||A||_F = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

The *p-norms*[3]

The matrix *p-norms* for $p = 1, 2$ and ∞ are defined as follows.

$$||A||_1 = \max_{j=1,\dots,n} \sum_{i=1}^{m} |a_{ij}| \quad \text{(the maximum absolute column sum)}$$

$$||A||_\infty = \max_{i=1,\dots,m} \sum_{j=1}^{n} |a_{ij}| \quad \text{(the maximum absolute row sum)}$$

$$||A||_2 = \sqrt{\lambda_{max}(A^T A)} \quad \text{(spectral norm)}$$

where $\lambda_{max}(A^T A)$ is the largest eigenvalue of $A^T A$.

Example:

Let $A = \begin{pmatrix} -2 & 3 \\ 0 & -4 \end{pmatrix}$.

Then $A^T A = \begin{pmatrix} -2 & 0 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} 4 & -6 \\ -6 & 25 \end{pmatrix}$

and the eigenvalues are $\lambda_1 = 26.59$ and $\lambda_2 = 2.41$ so that

$$||A||_1 = \max\{|-2| + |0|, |3| + |-4|\} = 7$$

$$||A||_\infty = \max\{|-2| + |3|, |0| + |-4|\} = 5$$

$$||A||_2 = \sqrt{\max\{26.59, 2.41\}} = 5.16$$

$$||A||_F = \sqrt{(-2)^2 + 3^2 + 0^2 + (-4)^2} = 5.39$$

2.2 Condition number

Definition 2.2. *[9, 12]. The condition number of a nonsingular matrix A is given by*

$$k(A) = ||A||\ ||A^{-1}||.$$

An alternative notation for the condition number of a matrix A is $cond(A)$. Different norms can be used to evaluate the condition number of a matrix. For this reason, the notation is usually modified to indicate the norm being used, for example if we use the infinity norm, then we can write

$$k_\infty(A) = ||A||_\infty\ ||A^{-1}||_\infty.$$

For every invertible matrix $A \in M_n(\mathbb{C})$, the following properties of the condition number hold:

1. $k(A) \geq 1$
2. $k(A) = k(A^{-1})$
3. $k(\alpha A) = k(A)$ for all $\alpha \in \mathbb{C} \setminus \{0\}$.

See e.g. [10].

Following the definition of condition number, one can easily compute the condition number of a matrix as illustrated in the example below.

Example:
Use the norm indicated to calculate the condition number of the following matrices.

i. $A = \begin{pmatrix} 2 & 3 \\ 4 & 6.1 \end{pmatrix}$; 1-norm

ii. $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & -3 \end{pmatrix}$; infinity-norm

Solution

i. $A = \begin{pmatrix} 2 & 3 \\ 4 & 6.1 \end{pmatrix}$
$||A||_1 = \max(2+4, 3+6.1) = 9.1,$

$$A^{-1} = \begin{pmatrix} 30.5 & -15 \\ -20 & 10 \end{pmatrix}$$

$$\Rightarrow ||A^{-1}||_1 = \max(30.5+20, 15+10) = 50.5.$$

Therefore $k_1(A) = ||A||_1\ ||A^{-1}||_1 = 9.1 \times 50.5 = 459.55$

ii. $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & -3 \end{pmatrix}$

$||B||_1 = \max(1+1+1, 1+1+2, 2+0+3) = 5,$

$$B^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{7}{12} & -\frac{5}{12} & -\frac{1}{12} \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \end{pmatrix}$$

$$\Rightarrow ||B^{-1}||_1 = \max\left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4}, \frac{7}{12} + \frac{5}{12} + \frac{1}{12}, \frac{1}{6} + \frac{1}{6} + \frac{1}{6}\right) = \frac{13}{12}.$$

Therefore $k_\infty(B) = ||B||_\infty \, ||B^{-1}||_\infty = 5 \times \frac{13}{12} = \frac{65}{12}$.

2.3 Conditioning

Consider the linear system

$$\begin{pmatrix} 2 & 3 \\ 4 & 6.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{2.1}$$

with the solution $x = \begin{pmatrix} 16 & -10 \end{pmatrix}^T$. If we perturb slightly the left hand side and have

$$\begin{pmatrix} 2 & 3 \\ 4.1 & 6.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{2.2}$$

then our new solution becomes $x = \begin{pmatrix} -32 & 22 \end{pmatrix}^T$ which is very far from the solution of (2.1). Again, system (2.3)

$$\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{2.3}$$

obtained by simply changing the coefficient 6.1 in (2.1) to 6 results in the total loss of the solution. This is a drastic change in the solution and we say that the system is very sensitive to small perturbations in the inputs. A system of this nature is called **ill-conditioned**. If small perturbations in the inputs of a linear system lead to a small change in the solution, then the system is called **well-conditioned**, otherwise it is **ill-conditioned**. This property of a linear system is termed the "conditioning" of the system and is determined by the **condition number** of the associated matrix. When the condition number of a matrix A is low, then the problem involving A is well-conditioned, otherwise it is ill-conditioned, see e.g. [3, 9, 10, 11].

3 Perturbation theory

In this section, we demonstrate the validity of perturbation theory by applying it to solve the algebraic equation $x^2 - 3x + 2 + \varepsilon = 0$ given by (1.1) and the harmonic oscillator equation $\ddot{x} + \varepsilon\dot{x} + x = 0$ given by (1.3). We shall obtain the exact solutions of their respective unperturbed equations and compare them with the perturbation solutions. With their graphs plotted, it will be easy for one to see how well the method of perturbation approximates solutions.

3.1 Algebraic equation

To solve the algebraic equation

$$x^2 - 3x + 2 + \varepsilon = 0, \quad 0 < \varepsilon \ll 1,$$

we assume that the roots have the following expansion:

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \tag{3.1}$$

Then by substitution we have:

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^2 - 3(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) + 2 + \varepsilon = 0.$$

Simplifying further and dropping higher powers of ε gives:

$$x_0^2 + 2\varepsilon x_0 x_1 + 2\varepsilon^2 x_0 x_2 + \varepsilon^2 x_1^2 - 3x_0 - 3\varepsilon x_1 - 3\varepsilon^2 x_2 + 2 + \varepsilon + \cdots = 0.$$

Collecting the powers of ε together gives:

$$(x_0^2 - 3x_0 + 2) + \varepsilon(2x_0 x_1 - 3x_1 + 1) + \varepsilon^2(2x_0 x_1 + x_1^2 - 3x_2) + \cdots = 0.$$

Equating the coefficient of each power of ε to zero gives:
ε^0 : $x_0^2 - 3x_0 + 2 = 0 \implies x_0 = 1$ or $x_0 = 2$.
ε^1 : $2x_0 x_1 - 3x_1 + 1 = 0$
from which we have $x_1 = 1$ for $x_0 = 1$ and $x_1 = -1$ for $x_0 = 2$.
ε^2 : $x_1^2 + 2x_0 x_1 - 3x_2 = 0$
from which we have $x_2 = 1$ for $x_0 = 1$ and $x_1 = 1$; and $x_2 = -1$ for $x_0 = 2$ and $x_1 = -1$.

Thus, $(x_0, x_1, x_2) = (1, 1, 1)$ or $(2, -1, -1)$.

By substituting these values into (3.1) we have:

$$x = 1 + \varepsilon + \varepsilon^2 + \cdots \quad \text{or} \quad x = 2 - \varepsilon - \varepsilon^2 + \cdots.$$

We could solve $x^2 - 3x + 2 + \varepsilon = 0$ with the quadratic formula to obtain:

$$x = \frac{3 \pm \sqrt{1 - 4\varepsilon}}{2} \tag{3.2}$$

and we know from Taylor's series that $\sqrt{1 - 4\varepsilon} = 1 - 2\varepsilon - 2\varepsilon^2 + \cdots$ so that by substituting this into (3.2) we have:

$$x = \frac{3 \pm (1 - 2\varepsilon - 2\varepsilon^2 + \cdots)}{2}$$

$$\implies x = 1 + \varepsilon + \varepsilon^2 + \cdots \quad \text{or} \quad x = 2 - \varepsilon - \varepsilon^2 + \cdots$$

which corresponds with the above solution.

Now, if we find the exact solution of the unperturbed equation $x^2 - 3x + 2 = 0$, we will obtain that $x = 1$ or $x = 2$.

Comparing the perturbation solution and the exact solution, one sees that if we set $\varepsilon = 0$, the two solutions become the same. For small ε, $0 < \varepsilon \ll 1$ as supposed, the perturbation solution only deviates slightly from the exact solution. For example, if we set $\varepsilon = 0.01$, then we have $x = 1.0101$ or $x = 1.9899$.

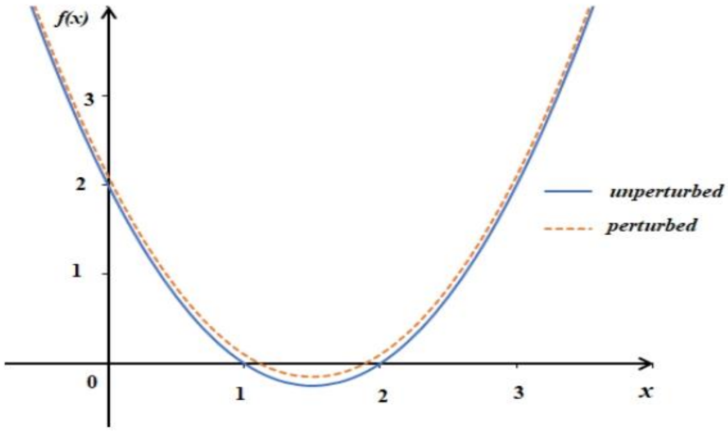

Figure 1: Graph of perturbed and unperturbed algebraic equation

From the graph, we see that for small ε, $0 < \varepsilon \ll 1$, the solution to the unperturbed equation given by (1.2) is a good approximation to that of the perturbed given by (1.1).

3.2 Damped harmonic oscillator

Here, we approximate the damped harmonic oscillator equation $\ddot{x} + \varepsilon\dot{x} + x = 0$ given by (1.3) by perturbation method using the exact solution of the undamped harmonic oscillator $\ddot{x} + x = 0$ given by (1.4).

Suppose we have the initial value problem

$$\ddot{x} + \varepsilon\dot{x} + x = 0 \quad ; \quad x(0) = 0 \quad , \quad \dot{x}(0) = 1.$$

Then by perturbation, we assume that the solution has the following expansion:

$$x(t) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \tag{3.3}$$

So that

$$\dot{x}(t) = \dot{x}_0 + \varepsilon\dot{x}_1 + \varepsilon^2\dot{x}_2 + \cdots$$
$$\ddot{x}(t) = \ddot{x}_0 + \varepsilon\ddot{x}_1 + \varepsilon^2\ddot{x}_2 + \cdots .$$

By substituting these into the given initial value problem, we have:

$$(\ddot{x}_0 + \varepsilon\ddot{x}_1 + \varepsilon^2\ddot{x}_2 + \cdots) + \varepsilon(\dot{x}_0 + \varepsilon\dot{x}_1 + \varepsilon^2\dot{x}_2 + \cdots) + x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots = 0.$$

Collecting the powers of ε together gives:

$$(\ddot{x}_0 + x_0) + \varepsilon(\dot{x}_0 + x_1 + \ddot{x}_1) + \varepsilon^2(\ddot{x}_2 + \dot{x}_1 + x_2) + \cdots = 0.$$

Equating the coefficient of each power of ε to zero gives:

Coefficients of ε^0 : $\ddot{x}_0 + x_0 = 0.$

By solving the characteristic equation $r^2 + 1 = 0$, we have that

$$r = \pm i$$

$$\implies x_0 = A\cos t + B\sin t$$

$$\dot{x}_0 = -A\sin t + B\cos t.$$

Applying the initial conditions gives:

$$x(0) = 0 \implies A = 0$$

$$\dot{x}(0) = 1 \implies B = 1.$$

Therefore,

$$x_0 = \sin t \tag{3.4}$$

$$\implies \dot{x}_0 = \cos t.$$

Coefficients of ε^1 : $\dot{x}_0 + x_1 + \ddot{x}_1 = 0$

$$\implies \cos t + x_1 + \ddot{x}_1 = 0$$

$$\ddot{x}_1 + x_1 = -\cos t \tag{3.5}$$

from which one quickly sees that the complementary function is $x_{1_h} = A\cos t + B\sin t$.

For the particular integral, we let

$$x_{1_p} = Ct\cos t + Dt\sin t$$

$$\implies \dot{x}_{1_p} = -Ct\sin t + C\cos t + Dt\cos t + D\sin t$$

$$\ddot{x}_{1_p} = -Ct\cos t - 2C\sin t - Dt\sin t + 2D\cos t.$$

By substituting these values into (3.5), we have:

$$-Ct\cos t - 2C\sin t - Dt\sin t + 2D\cos t + Ct\cos t + Dt\sin t = -\cos t$$

$$\implies 2D\cos t - 2C\sin t = -\cos t.$$

By comparing coefficients, we have:

$$C = 0 \text{ and } D = -\frac{1}{2}.$$

And so,

$$x_{1_p} = -\frac{1}{2}t\sin t.$$

But

$$x_1 = x_{1_h} + x_{1_p}.$$

Therefore,

$$x_1 = A\cos t + B\sin t - \frac{1}{2}t\sin t$$

$$\dot{x}_1 = -A\sin t + B\cos t - \frac{1}{2}t\cos t - \frac{1}{2}\sin t.$$

Applying the initial conditions gives:

$$x_1 = \sin t - \frac{1}{2}t\sin t \tag{3.6}$$

$$\implies \dot{x}_1 = \cos t - \frac{1}{2}t\cos t - \frac{1}{2}\sin t.$$

Coefficients of ε^2 : $\ddot{x}_2 + \dot{x}_1 + x_2 = 0$

$$\implies \ddot{x}_2 + \cos t - \frac{1}{2}t\cos t - \frac{1}{2}\sin t + x_2 = 0$$

$$\ddot{x}_2 + x_2 = \frac{1}{2}t\cos t + \frac{1}{2}\sin t - \cos t \tag{3.7}$$

from which we have the complementary function $x_{2_h} = A\cos t + B\sin t$.

For the particular integral, we assume

$$x_{2_p} = (Ct^2 + Dt)\cos t + (Et^2 + Ft)\sin t$$

$$\implies \dot{x}_{2_p} = -(Ct^2 + Dt)\sin t + (2Ct + D)\cos t + (Et^2 + Ft)\cos t + (2Et + F)\sin t$$

$$\ddot{x}_{2_p} = -(Ct^2 + Dt)\cos t - (2Ct + D)\sin t - (2Ct + D)\sin t + 2C\cos t - (Et^2 + Ft)\sin t$$
$$+(2Et + F)\cos t + (2Et + F)\cos t + 2E\sin t.$$

By substituting these values into (3.7), we have:

$$-(Ct^2 + Dt)\cos t - (2Ct + D)\sin t - (2Ct + D)\sin t + 2C\cos t - (Et^2 + Ft)\sin t + (2Et + F)\cos t$$
$$+(2Et + F)\cos t + 2E\sin t + (Ct^2 + Dt)\cos t + (Et^2 + Ft)\sin t = \frac{1}{2}t\cos t + \frac{1}{2}\sin t - \cos t$$

which simplifies to

$$-4Ct\sin t + 4Et\cos t + (-2D + 2E)\sin t + (2C + 2F)\cos t = \frac{1}{2}t\cos t + \frac{1}{2}\sin t - \cos t.$$

By comparing coefficients, we have:

$$C = 0, \quad D = -\frac{1}{8}, \quad E = \frac{1}{8} \text{ and } F = -\frac{1}{2}.$$

And so,

$$x_{2_p} = -\frac{1}{8}t\cos t + (\frac{1}{8}t^2 - \frac{1}{2}t)\sin t.$$

But

$$x_2 = x_{2_h} + x_{2_p}.$$

Therefore,

$$x_2 = A\cos t + B\sin t - \frac{1}{8}t\cos t + (\frac{1}{8}t^2 - \frac{1}{2}t)\sin t$$

$$\dot{x}_2 = -A\sin t + B\cos t + \frac{1}{8}t\sin t - \frac{1}{8}\cos t + (\frac{1}{8}t^2 - \frac{1}{2}t)\cos t + (\frac{1}{4}t - \frac{1}{2})\sin t.$$

Applying the initial conditions gives:

$$x_2 = \sin t - \frac{1}{8}t\cos t + (\frac{1}{8}t^2 - \frac{1}{2}t)\sin t \tag{3.8}$$

$$\implies \dot{x}_2 = \cos t + \frac{1}{8}t\sin t - \frac{1}{8}\cos t + (\frac{1}{8}t^2 - \frac{1}{2}t)\cos t + (\frac{1}{4}t - \frac{1}{2})\sin t.$$

So far, we have from (3.4), (3.6) and (3.8) the following solutions:

$$x_0 = \sin t$$
$$x_1 = \sin t - \frac{1}{2}t\sin t$$
$$x_2 = \sin t - \frac{1}{8}t\cos t + (\frac{1}{8}t^2 - \frac{1}{2}t)\sin t.$$

Substituting these values into (3.3) gives:

$$x(t) = \sin t + \varepsilon(\sin t - \frac{1}{2}t\sin t) + \varepsilon^2[\sin t - \frac{1}{8}t\cos t + (\frac{1}{8}t^2 - \frac{1}{2}t)\sin t] + \cdots.$$

The unperturbed problem

$$\ddot{x} + x = 0 \quad ; \quad x(0) = 0 \quad , \quad \dot{x}(0) = 1$$

has the exact solution

$$x(t) = \sin t.$$

Comparing the perturbation solution and the exact solution, one sees that if we set $\varepsilon = 0$, the two solutions become the same.

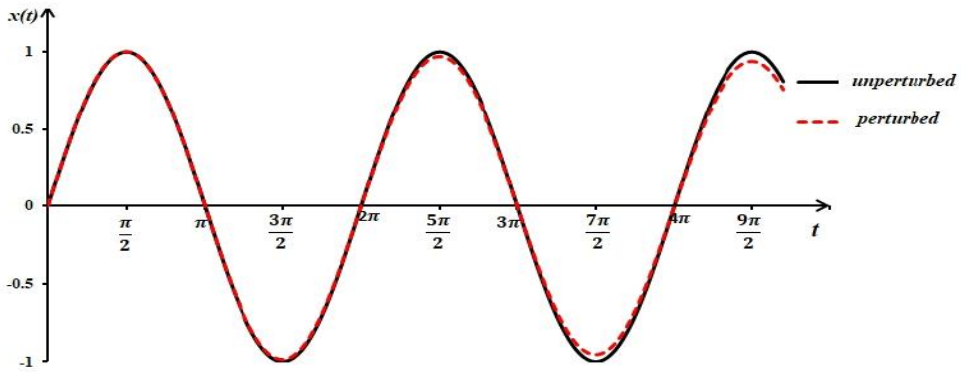

Figure 2: Damped and undamped harmonic oscillator equation for $0 \leq t \leq \frac{9}{2}\pi$

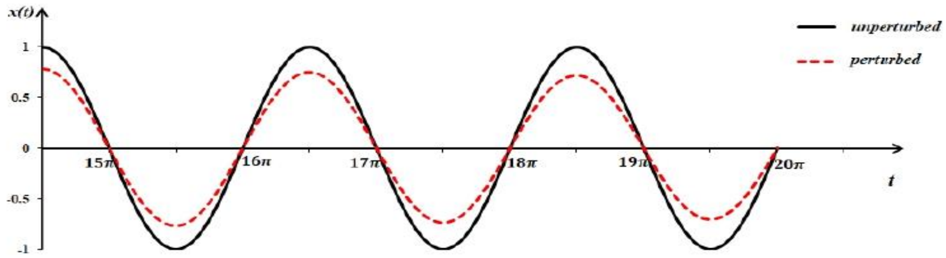

Figure 3: Damped and undamped harmonic oscillator equation for $\frac{29}{2}\pi \leq t \leq 20\pi$

From the graphs, we observe that the exact solution of the unperturbed problem (i.e. undamped harmonic oscillator equation) given by (1.4) is a good approximation to the perturbed one (i.e. damped harmonic oscillator equation) given by (1.3) for a considerable period of time. For example, on the interval $0 \leq t \leq \frac{9}{2}\pi$, the deviation of the perturbation solution from the exact one is insignificant. But if one waits long enough, for example, on the interval $\frac{29}{2}\pi \leq t \leq 20\pi$, the deviation becomes appreciable. This is due to the fact that the amplitude of the damped one keeps decreasing with time while that of the undamped remains constant. Hence, we conclude that the method of perturbation is an effective tool for handling problems of this nature which are more often than not encountered in real life.

4 Perturbation of linear systems

In this section, we examine the effects of small perturbations in the coefficient matrices of linear systems to the solution vectors. It is known that some systems are very sensitive in the sense that small perturbations in the coefficient matrices trigger off large changes in the solution vectors whereas the reverse is the case for some others, see e.g. [11, 3]. We will find out how this is connected to the matrix norms of the associated matrices. We begin with systems of algebraic equations and then move on to systems of ordinary differential equations which form an integral part of this study.

4.1 Systems of algebraic equations

Consider the linear system

$$\begin{pmatrix} 40 & 20.1 \\ 80 & 40.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 241 \\ 481 \end{pmatrix}$$

whose solution vector is $x = \begin{pmatrix} 1 & 10 \end{pmatrix}^T$. Now, suppose we have the perturbed system

$$\begin{pmatrix} 40+\varepsilon & 20.1 \\ 80 & 40.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 241 \\ 481 \end{pmatrix}$$

and set $\varepsilon = 0.1$, then the resulting system becomes

$$\begin{pmatrix} 40.1 & 20.1 \\ 80 & 40.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 241 \\ 481 \end{pmatrix}$$

with the solution vector $x = \begin{pmatrix} -400 & 810 \end{pmatrix}^T$. This is a drastic change in the solution vector. Hence, we say that the solution is very sensitive to small perturbations in the coefficient matrix.

Now, we compute the condition number of the matrix $A = \begin{pmatrix} 40 & 20.1 \\ 80 & 40.1 \end{pmatrix}$ and see its relationship with this high sensitivity.

Using the 1-norm, we have:

$$||A||_1 = \max(40 + 80, 20.1 + 40) = 120$$

$$A^{-1} = \begin{pmatrix} -10.025 & 5.025 \\ 20 & -10 \end{pmatrix}$$

$$\implies ||A^{-1}||_1 = \max(10.025 + 20, 5.025 + 10) = 30.025.$$

Therefore, $k_1(A) = ||A||_1 \; ||A^{-1}||_1 = 120 \times 30.025 = 3603$

Thus, the condition number is large and the solution is very sensitive.

Consider also the linear system

$$\begin{pmatrix} 0 & 40 \\ 50 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 40 \\ 49 \end{pmatrix}.$$

This system has the solution vector $x = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$. Suppose we have the perturbed system $\begin{pmatrix} \varepsilon & 40 \\ 50 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 40 \\ 49 \end{pmatrix}$ and set $\varepsilon = 0.1$, then the resulting system becomes

$$\begin{pmatrix} 0.1 & 40 \\ 50 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 40 \\ 49 \end{pmatrix}$$

with the solution vector $x = \begin{pmatrix} 0.99995 & 0.99750 \end{pmatrix}^T$. In this case, we see that the solution is not very sensitive to small changes made in the coefficient matrix.

Now, we compute the condition number of the matrix $A = \begin{pmatrix} 0 & 40 \\ 50 & -1 \end{pmatrix}$ using the 1-norm, as follows:

$$||A||_1 = \max(0 + 50, 40 + 1) = 50$$

$$A^{-1} = \begin{pmatrix} 0.0005 & 0.02 \\ 0.025 & 0 \end{pmatrix}$$

$$\implies ||A^{-1}||_1 = \max(0.0005 + 0.025, 0.02 + 0) = 0.0255.$$

Therefore, $k_1(A) = ||A||_1 \; ||A^{-1}||_1 = 50 \times 0.0255 = 1.275$

Thus, the condition number is small and the solution is not very sensitive.

4.2 Systems of ordinary differential equations

Consider the linear system of first order ordinary differential equations

$$\dot{x} = \begin{pmatrix} 4 & 2.1 \\ 8 & 4.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

whose general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1.9281 \end{pmatrix} e^{-0.0491t} + c_2 \begin{pmatrix} 1 \\ 1.9756 \end{pmatrix} e^{8.1491t}.$$

Suppose we have the perturbed system $\dot{x} = \begin{pmatrix} 4+\varepsilon & 2.1 \\ 8 & 4.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and set $\varepsilon = 0.1$, then the resulting system becomes

$$\dot{x} = \begin{pmatrix} 4.1 & 2.1 \\ 8 & 4.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

with the general solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1.9518 \end{pmatrix} e^{8.1988t} + c_2 \begin{pmatrix} -1 \\ 1.9518 \end{pmatrix} e^{0.0012t}.$$

And we see that the solution is very sensitive to small perturbations in the coefficient matrix.

We compute the condition number of the matrix $A = \begin{pmatrix} 4 & 2.1 \\ 8 & 4.1 \end{pmatrix}$ using the 1-norm as follows:

$$||A||_1 = \max(4+8, 2.1+4) = 12$$

$$A^{-1} = \begin{pmatrix} -10.25 & 5.25 \\ 20 & -10 \end{pmatrix}$$

$$\implies ||A^{-1}||_1 = \max(10.25+20, 5.25+10) = 30.25.$$

Therefore, $k_1(A) = ||A||_1 \; ||A^{-1}||_1 = 12 \times 30.25 = 363$
Thus, the condition number is large and the solution is very sensitive.

Now, we consider the unperturbed linear system

$$\dot{x} = \begin{pmatrix} 0 & 4 \\ 5 & -0.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

described by (1.6) above whose general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1.1056 \end{pmatrix} e^{4.4224t} + c_2 \begin{pmatrix} -1 \\ 1.1306 \end{pmatrix} e^{-4.5224t}.$$

Also, consider the perturbed system $\dot{x} = \begin{pmatrix} \varepsilon & 4 \\ 5 & -0.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ described by (1.5) above. If we set $\varepsilon = 0.1$, then the resulting system becomes

$$\dot{x} = \begin{pmatrix} 0.1 & 4 \\ 5 & -0.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

with the general solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1.0933 \end{pmatrix} e^{4.4733t} + c_2 \begin{pmatrix} -1 \\ 1.1433 \end{pmatrix} e^{-4.4733t}.$$

And we see that the solution is not very sensitive to small perturbations in the coefficient matrix.

We compute the condition number of the matrix $A = \begin{pmatrix} 0 & 4 \\ 5 & -0.1 \end{pmatrix}$ using the 1-norm as follows:

$$||A||_1 = \max(0 + 5, 4 + 0.1) = 5$$

$$A^{-1} = \begin{pmatrix} 0.005 & 0.2 \\ 0.25 & 0 \end{pmatrix}$$

$$\implies ||A^{-1}||_1 = \max(0.005 + 0.25, 0.2 + 0) = 0.255.$$

Therefore, $k_1(A) = ||A||_1 \ ||A^{-1}||_1 = 5 \times 0.255 = 1.275$
Thus, the condition number is small and the solution is not very sensitive.

5 Conclusion

We have discussed the concepts of matrix norms, condition numbers, perturbation theory and their applications in sensitivity analysis of the solutions of perturbed linear systems. At first, we found that the method of perturbation is a powerful tool for finding an approximate analytical solution to both algebraic and differential equations. The results obtained by plotting the graphs of the perturbation and exact solutions of both the algebraic and differential equations show a negligible difference between the two equations. Although, for the harmonic oscillator equation, the difference becomes appreciable over time due to the fact that the amplitude of the damped one keeps decreasing with time while that of the undamped remains constant.

Furthermore, the study indicates that the systems of equations whose associated matrices have large condition numbers are usually very sensitive to small perturbations while the reverse is the case for low condition numbers which is in conformity with previous studies.

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